

# Abelian geometric phase for a Dirac neutral particle in a Lorentz symmetry violation environment

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## Abstract

We introduce a new term into the Dirac equation based on the Lorentz symmetry violation background in order to make a theoretical description of the relativistic quantum dynamics of a spin-half neutral particle, where the wave function of the neutral particle acquires a relativistic Abelian quantum phase given by the interaction between a fixed time-like 4-vector background and crossed electric and magnetic fields, which is analogous to the geometric phase obtained by Wei *et al* [H. Wei, R. Han and X. Wei, Phys. Rev. Lett. **75**, 2071 (1995)] for a spinless neutral particle with an induced electric dipole moment. We also discuss the flux dependence of energy levels of bound states analogous to the Aharonov-Bohm effect for bound states.

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## I. INTRODUCTION

In recent years, the appearance of topological or geometric phases has attracted a great deal of interest in several physical systems such as in condensed matter [1], in cosmological backgrounds [2] and for scalar quantum particles in spacetimes generated by defects [3]. The Aharonov-Bohm effect [4] is the best famous topological effect which has been studied in scattering problems, and in the flux dependence of the energy levels of bound states [5]. Another interesting study of topological effects in quantum systems is the dual effect of the Aharonov-Bohm effect made by Dowling *et al* [6], and Furtado *et al* [7]. A well-known quantum effect related to the appearance of quantum phases is the Aharonov-Casher effect [8], which can be considered as the reciprocal effect of the Aharonov-Bohm effect. More discussions about topological quantum effects can be found in the literature as the equivalence between the Aharonov-Bohm and the Aharonov-Casher effects [9, 10], the topological nature of the Aharonov-Casher effect [11, 12], the nonlocality and the nondispersivity [13–17]. Other studies based on the Aharonov-Casher system have been done in the Landau quantization [18–21], bound states analogous to a quantum dot induced by noninertial effects [22], the appearance of geometric phases in the noncommutative quantum mechanics [23], and in the presence of topological defects [24].

The dual effect of the Aharonov-Casher effect has been proposed by He and McKellar [25], and Wilkens [26], which is known as the He-McKellar-Wilkens effect. Involving a non-constant electric dipole moment, Wei *et al* [27] proposed a new system where the wave function of a neutral particle acquires a quantum phase when the induced electric dipole moment interacts with a configuration of crossed electric and magnetic fields. Several discussions have been done about the topological nature of the quantum phases for electric dipoles [28–34], and experiments have also been proposed with the purpose of producing the He-McKellar-Wilkens setup [35–39]. The solid state analogue of the He-McKellar-Wilkens quantum phase has been discussed in [40]. Furthermore, bound states for a neutral particle with an induced electric dipole moment have been studied in the Landau quantization [41], and in the presence of an azimuthal magnetic field and in the presence of a disclination [42].

Recently, general discussions about quantum effects involving magnetic and electric dipole moments have been done in the nonrelativistic context, and in the relativistic context of quantum mechanics [43–47]. For instance, the relativistic quantum dynamics of a neutral

particle with permanent magnetic dipole moment can be described by introducing a non-minimal coupling into the Dirac equation given by  $i\gamma^\mu \partial_\mu \rightarrow i\gamma^\mu \partial_\mu + \frac{\mu}{2} \Sigma^{\mu\nu} F_{\mu\nu}$  [43–46], where  $\mu$  is the permanent magnetic dipole moment of the neutral particle,  $F_{\mu\nu}$  is the electromagnetic tensor and  $\Sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu]$ , with  $\gamma^\mu$  being the Dirac matrices. For a neutral particle with permanent electric dipole moment, the relativistic behavior has been introduced in the Dirac equation by Salpeter [47]. In recent works [43–45], the relativistic behaviour of a neutral particle with permanent electric dipole moment can be described by the dual transformation  $\mu \leftrightarrow d$ ,  $\vec{E} \leftrightarrow -\vec{B}$  and  $\vec{B} \leftrightarrow \vec{E}$  [45], where the nonminimal coupling becomes  $i\gamma^\mu \partial_\mu \rightarrow i\gamma^\mu \partial_\mu - i\frac{d}{2} \Sigma^{\mu\nu} \gamma^5 F_{\mu\nu}$  [43–45, 47], with  $d$  being the permanent electric dipole moment of the neutral particle. Other studies of the quantum dynamics of a neutral particle with a magnetic dipole moment have been done based on the violation of the Lorentz symmetry [48–51]. In Refs. [48–51], the quantum dynamics of a neutral particle is described by introducing a nonminimal coupling into the Dirac equation given by  $i\gamma^\mu \partial_\mu \rightarrow i\gamma^\mu \partial_\mu - g v^\nu \tilde{F}_{\mu\nu} \gamma^\mu$ , where  $v^\nu$  is a fixed 4-vector which acts as the background which breaks the Lorentz symmetry,  $\tilde{F}_{\mu\nu}$  corresponds to the dual electromagnetic tensor, and  $g$  is a coupling constant. One should note that, for relativistic models, the study of symmetry breaking can be extended by considering a background given by a 4-vector field that breaks the symmetry  $\mathcal{SO}(1, 3)$  instead of the symmetry  $\mathcal{SO}(3)$ . This is known in the literature as the spontaneous violation of the Lorentz symmetry [52–54]. The spontaneous violation of Lorentz symmetry was first proposed in 1989 by Kostelecky and Samuel [52] indicating the possibility that the spontaneous violation of symmetry by a scalar field could be extended to the string field theory. The consequence of this extension is a spontaneous breaking of the Lorentz symmetry as in the electroweak theory. In the electroweak theory, a scalar field acquires a nonzero vacuum expectation value that yields mass to gauge bosons (Higgs Mechanism). In a similar way, one can find in the string theory that this scalar field can be extended to a tensor field. At present days, theories involving the Lorentz symmetry breaking are part of the proposal of the Extended Standard Model [55] as a possible extension of the minimal Standard Model of the fundamental interactions, where the violation of the Lorentz symmetry is implemented in the fermion section of the Extended Standard Model by two CPT-odd terms, that is,  $v_\mu \bar{\psi} \gamma^\mu \psi$  and  $b_\mu \bar{\psi} \gamma_5 \gamma^\mu \psi$  (where  $v_\mu$  and  $b_\mu$  correspond to the Lorentz-violating vector field backgrounds [48–50, 56]). Furthermore, the modified Dirac theory has already been examined in [57], and the spectrum of energy of the hydrogen atom

has been discussed in the nonrelativistic limit of the modified Dirac theory in [58, 59]. We can also find in the literature an extensive amount of work looking at the violation of the Lorentz symmetry, and numerous experimental bounds were estimated [54]. However, is the introduction of this new nonminimal coupling  $i\gamma^\mu \partial_\mu \rightarrow i\gamma^\mu \partial_\mu - g v^\nu \tilde{F}_{\mu\nu} \gamma^\mu$  consistent with studies of the Lorentz violation symmetry? We will adopt the following point of view: a nonminimal coupling is conceived as corrections of a dynamical field, thus, the theory goes on to describe a new behavior, when the physical system begin to access a new energy scale. Such coupling with the background can present relevant information about the low energy regime of a fundamental theory. Hence, if we have a fundamental theory with vector fields (or tensor) which violate the Lorentz symmetry, by assuming non-trivial expected values in vacuum, we propose that the fermion sector should feels this background by a nonminimal coupling with a vector background [48–50]. Specifically, in our case, such coupling implies that the background can generate a Aharonov-Casher-type phase to a neutral particle.

In this paper, based on the introduction of a nonminimal coupling into the Dirac equation to describe the quantum dynamics of a neutral particle in the Lorentz symmetry violation background [48–51], we suggest the introduction of a new term into the Dirac equation based on the Lorentz symmetry violation background to describe the relativistic quantum dynamics of a spin-half neutral particle, where the wave function of the neutral particle acquires a relativistic Abelian quantum phase analogous to the geometric phase obtained by Wei *et al* [27] for an induced electric dipole moment. We show that the relativistic geometric phase differs from the relativistic Aharonov-Casher geometric phase obtained in Ref. [51] based on the Lorentz symmetry violation background, since the relativistic geometric phase obtained in this work is given by the interaction between a fixed time-like 4-vector background and crossed electric and magnetic fields. At the end, we also show that the energy levels for bound states of a spin-half neutral particle confined to moving between two coaxial cylinders are flux dependent. The structure of this paper is: in section II, we introduce a new term in the Dirac equation based on the Lorentz symmetry violation background, and discuss the relativistic analogue of the geometric quantum phase obtained by Wei *et al* [27] for a spin-half neutral particle, which is given by the interaction between a fixed time-like 4-vector background and crossed electric and magnetic fields. We also discuss the flux dependence of the relativistic energy levels for bound states; in section III, we present our conclusions.

## II. RELATIVISTIC ABELIAN GEOMETRIC PHASE BASED ON THE LORENTZ SYMMETRY BREAKING

In this section, we suggest the introduction of a new term into the Dirac equation based on the Lorentz symmetry violation background to describe a relativistic quantum system, where the wave function of the neutral particle acquires a relativistic quantum phase analogous to the geometric phase obtained by Wei *et al* [27] in the rest frame of the neutral particle. Furthermore, we discuss the nonrelativistic limit of the Dirac equation, and obtain a geometric quantum phase for a spin-half particle analogous to that one obtained by Wei *et al* [27]. At the end of the section, we discuss the flux dependence of the energy levels for a relativistic neutral particle confined to moving between two coaxial cylinders. In that way, we introduce this new term in the Dirac equation as follows

$$i\gamma^\mu \partial_\mu \rightarrow i\gamma^\mu \partial_\mu - \frac{g}{2} \eta^{\alpha\beta} F_{\mu\alpha} F_{\beta\nu} \gamma^\mu b_\lambda \gamma^\lambda \gamma^\nu - \frac{\nu}{2} \eta^{\alpha\beta} F_{\mu\alpha} F_{\beta\nu} \gamma^\mu \gamma^\nu, \quad (1)$$

where  $g$  and  $\nu$  are constants,  $b_\lambda$  corresponds to the fixed 4-vector that breaks the Lorentz symmetry,  $F_{\mu\nu}$  is the electromagnetic field tensor whose components are  $F_{0i} = -F_{i0} = -E_i$ ,  $F_{ij} = -F_{ji} = \epsilon_{ijk} B^k$ , and the matrices  $\gamma^\mu$  are the Dirac matrices given in the Minkowski spacetime [46, 60], *i.e.*,

$$\gamma^0 = \hat{\beta} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}; \quad \gamma^i = \hat{\beta} \hat{\alpha}^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}; \quad \Sigma^i = \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}, \quad (2)$$

with  $I$  being the  $2 \times 2$  identity matrix,  $\vec{\Sigma}$  being the spin vector, and  $\sigma^i$  being the Pauli matrices. The Pauli matrices satisfy the relation  $(\sigma^i \sigma^j + \sigma^j \sigma^i) = 2\eta^{ij}$ , where  $\eta^{\mu\nu} = \text{diag}(-, +, +, +)$  is the Minkowski tensor. By taking the Lorentz symmetry violation background given by a time-like vector  $b^\lambda = (b^0, 0, 0, 0)$  in such a way that we can consider  $gb^0 = \nu$  [68], and using the definitions of  $F_{\mu\nu}$  and  $\Sigma^{\mu\nu}$  given earlier, we can develop the nonminimal coupling term of the expression (1) and obtain  $\frac{\nu}{2} \eta^{\alpha\beta} F_{\mu\alpha} F_{\beta\nu} \gamma^\mu (\gamma^0 + I) \gamma^\nu = \nu \hat{\beta} \vec{\alpha} \cdot (\vec{E} \times \vec{B}) - 2\nu B^2 \hat{\beta} - \nu (E^2 - B^2)$ . With direct calculations, we can verify that the new term introduced in the equation (1) is Hermitian, thus, we have a toy model to study relativistic geometric quantum phases for a spin-half neutral particle analogous to a neutral particle with an induced electric dipole moment based on the background where the Lorentz

symmetry is violated. We will show that we can make a study of relativistic quantum mechanics in this background, where the relativistic analogue of the geometric phase obtained by Wei *et al* [27] can be obtained and, in the nonrelativistic limit, we can obtain an analogous system to that worked by Wei *et al* [27] for a spinless neutral particle with an induced electric dipole moment.

Now, let us consider the same the field configuration given in Ref. [27]. In this way, the spin-half neutral particle interacts with a configuration of crossed electric and magnetic field given by

$$\vec{E} = \frac{\lambda}{\rho} \hat{\rho}; \quad \vec{B} = B_0 \hat{z}, \quad (3)$$

with  $\lambda$  being a linear density of electric charges and  $B_0$  being a constant. In this system, it is convenient to work with the cylindrical symmetry, where the line element of the Minkowski spacetime is written in the form  $ds^2 = -dt^2 + d\rho^2 + \rho^2 d\varphi^2 + dz^2$ . Since we are working with curvilinear coordinates, it is convenient to treat the Dirac spinors by using the mathematical formulation of spinor theory in curved spacetime background [61]. In curved spacetime background, the spinors are defined locally in the local reference frame of the observers [61]. We can build the local reference frame for the observers through a noncoordinate basis  $\hat{\theta}^a = e^a_{\mu}(x) dx^{\mu}$ , which components  $e^a_{\mu}(x)$  satisfy the following relation  $g_{\mu\nu}(x) = e^a_{\mu}(x) e^b_{\nu}(x) \eta_{ab}$  [61]. The indices  $(a, b, c = 0, 1, 2, 3)$  indicate the local reference frame of the observers and the indices  $(\mu, \nu = t, \rho, \varphi, z)$  indicate the spacetime indices. The components of the noncoordinate basis  $e^a_{\mu}(x)$  are called *tetrads* and they form the local reference frame of the observers. The tetrads have an inverse defined by  $dx^{\mu} = e^{\mu}_a(x) \hat{\theta}^a$ , where they are related by the expressions  $e^a_{\mu}(x) e^{\mu}_b(x) = \delta^a_b$  and  $e^{\mu}_a(x) e^a_{\nu}(x) = \delta^{\mu}_{\nu}$ . Moreover, in curved spacetime background, the partial derivative  $\partial_{\mu}$  must be changed by the covariant derivative of a spinor [61, 62] given by  $\nabla_{\mu} = \partial_{\mu} + \Gamma_{\mu}(x)$ , with  $\Gamma_{\mu}(x) = \frac{i}{4} \omega_{\mu ab}(x) \Sigma^{ab}$  being the spinorial connection [61, 62]. Choosing  $\hat{\theta}^0 = dt$ ,  $\hat{\theta}^1 = d\rho$ ,  $\hat{\theta}^2 = \rho d\varphi$  and  $\hat{\theta}^3 = dz$ , we can obtain the non-null components of 1-form connection  $\omega_{\mu ab}(x)$  by solving the Maurer-Cartan structure equations  $d\hat{\theta}^a + \omega^a_b \wedge \hat{\theta}^b = 0$  [62], with  $\omega^a_b = \omega^a_{\mu b}(x) dx^{\mu}$ . Direct calculations give us  $\omega_{\varphi}^1{}_2(x) = -\omega_{\varphi}^2{}_1(x) = -1$  and, consequently,  $\gamma^{\mu} \Gamma_{\mu} = \frac{\gamma^1}{2\rho}$  [20]. Hence, based on the nonminimal coupling (1), the Dirac equation becomes

$$i \frac{\partial \psi}{\partial t} = m \hat{\beta} \psi + \vec{\alpha} \cdot \left[ \vec{p} - i \vec{\xi} + \nu \vec{E} \times \vec{B} \right] \psi - \nu B^2 \psi - \nu \hat{\beta} (E^2 - B^2) \psi, \quad (4)$$

where we have defined the vector  $\vec{\xi}$  whose non-null components are  $-i\xi_k = -ie_k^\varphi(x)\Gamma_\varphi = -\frac{\Sigma^3}{2\rho}\delta_{2k}$ . Note that, given the field configuration (3), the operator  $\hat{\Pi} = \vec{p} - i\vec{\xi} + \nu \vec{E} \times \vec{B}$  is not a constant of motion [69], but there are no classical forces acting on the dipole moment due to the term  $\nu \vec{E} \times \vec{B}$ , which indicates that the relativistic quantum phase generated by the presence of the term  $\nu \vec{E} \times \vec{B}$  in the Dirac equation (4) has a topological nature in the same sense of that discussed in Refs. [8, 11, 27].

Let us first discuss the nonrelativistic limit of the Dirac equation (4), and show that we can obtain a nonrelativistic equation of motion for a spin-half neutral particle analogous to the nonrelativistic equation of motion for a spinless neutral particle with an induced electric dipole moment worked by Wei *et al.* in [27]. We can obtain the nonrelativistic dynamics of the neutral particle when we extract the temporal dependence of the wave function due to the rest energy [46, 60], thus, we write the Dirac spinor in the form  $\psi = e^{-imt} (\phi \chi)^T$ , where  $\phi$  and  $\chi$  are two-spinors. Substituting this solution into the Dirac equation (4), we obtain two coupled equations for  $\phi$  and  $\chi$ , where the first coupled equation is  $i\frac{\partial\phi}{\partial t} + \nu E^2 \phi = \left[ \vec{\sigma} \cdot (\vec{p} - i\vec{\xi}) + \nu \vec{\sigma} \cdot (\vec{E} \times \vec{B}) \right] \chi$ . By considering  $\phi$  being the “large” component and  $\chi$  being the “small” component, we can take  $|2m\chi| \gg |i\frac{\partial\chi}{\partial t}|$ ,  $|2m\chi| \gg \frac{\nu}{2}(E^2 + B^2)\chi$ , and write the second coupled equation in the form:  $\chi \approx \frac{1}{2(m+\nu B^2)} \left[ \vec{\sigma} \cdot (\vec{p} - i\vec{\xi}) + \nu \vec{\sigma} \cdot (\vec{E} \times \vec{B}) \right] \phi$  without loss of generality. In that way, substituting  $\chi$  of the second coupled equation into the first coupled equation and neglecting terms of order  $m^{-2}$ , we obtain the following second order differential equation

$$i\frac{\partial\phi}{\partial t} \approx \frac{1}{2(m+\nu B^2)} \left[ \vec{p} - i\vec{\xi} + \nu (\vec{E} \times \vec{B}) \right]^2 \phi - \nu E^2 \phi + \frac{\nu}{2(m+\nu B^2)} \vec{\sigma} \cdot \vec{B}_{\text{eff}} \phi, \quad (5)$$

which corresponds to the nonrelativistic limit of the Dirac equation (4). Note that, up to the term  $-i\vec{\xi}$  and the last term of the equation (5), we have the Schrödinger equation for a spinless neutral particle with an induced electric dipole moment worked in [27]. The last term of the nonrelativistic equation (5) appears due to the interaction between the spin and an effective magnetic field given by  $\vec{B}_{\text{eff}} = \vec{\nabla} \times (\vec{E} \times \vec{B})$ . Taking the field configuration (3), we can see that the effective magnetic field is null ( $B_{\text{eff}} = 0$ ), and the nonrelativistic equation (5) becomes the Schrödinger equation obtained in [27] (up to the term  $-i\vec{\xi}$ ). Moreover, we obtain in Eq. (5) the phase shift  $\Phi_{\text{NR}} = \oint \nu (\vec{E} \times \vec{B}) \cdot d\vec{r} = 2\pi \nu \lambda B_0$ , which corresponds to the Abelian geometric phase analogous to that obtained by Wei *et al* [27], but for a spin-half neutral particle, where we have obtained in the Lorentz symmetry violation background.

Hence, we have shown that the nonrelativistic limit of the Dirac equation (4) yields both the equation of motion and the Abelian geometric phase analogous to that obtained by Wei *et al* [27], but, in this case, for a spin-half neutral particle in the Lorentz symmetry violation background.

From now on, we will discuss the appearance of a relativistic geometric phase analogous to the geometric phase obtained by Wei *et al* [27], when we consider the introduction of the new term in the Dirac equation (1), and the field configuration given in (3). To obtain the relativistic quantum phase, we apply the Dirac phase factor method [63], where the wave function of the neutral particle is written in the form  $\psi = e^{i\Phi} \psi_0$ . Substituting this ansatz into the Dirac equation (4), we have that  $\psi_0$  is the solution of the equation

$$i\gamma^0 \frac{\partial \psi_0}{\partial t} + i\gamma^1 \left( \frac{\partial}{\partial \rho} + \frac{1}{2\rho} \right) \psi_0 + i\frac{\gamma^2}{\rho} \frac{\partial \psi_0}{\partial \varphi} + i\gamma^3 \frac{\partial \psi_0}{\partial z} + \nu E^2 \psi_0 - m\psi_0 = 0, \quad (6)$$

where the term proportional to  $E^2$  is a local term and does not contribute to the geometric phase [23, 43, 44]. We also have that the term proportional to  $B^2$  provides a contribution to the dynamical phase given by  $\Phi_D = -\nu \int_0^\tau (\hat{\beta} - I) B^2 dt = -\nu B_0^2 \tau \hat{\beta} - \nu B_0^2 \tau$ , which is equivalent to that one given in [27]. The relativistic geometric quantum phase acquired by the wave function of the neutral particle is

$$\Phi = -\nu \oint \left( \vec{E} \times \vec{B} \right)_\varphi d\varphi = 2\pi \nu \lambda B_0. \quad (7)$$

The relativistic quantum phase (7) is obtained by the interaction between a fixed time-like 4-vector and a configuration of crossed electric and magnetic fields given in (3). As discussed earlier, there are no classical forces acting on the dipole moment due to the presence of the term  $\nu \vec{E} \times \vec{B}$ . Thus, the relativistic quantum phase (7) has a topological nature [8, 11] in the same way of the nonrelativistic case given in [27]. Moreover, we can also see that the relativistic quantum phase is nondispersive [15–17], that is, it does not depend of the velocity of the neutral particle. In this sense, we can see that the expression (7) corresponds to the relativistic analogue of the geometric quantum phase obtained by Wei *et al* [27] in the Lorentz symmetry violation background. Furthermore, We can easily see that the relativistic geometric phase (7) differs from the relativistic Aharonov-Casher geometric phase based on the Lorentz symmetry violation background obtained in [51], since the relativistic geometric phase (7) is provided by the interaction between a time-like 4-vector background ( $\nu = g b^0$ ) and a field configuration given by crossed electric and magnetic fields. Thus, the relativistic



geometric phase (7) brings us a new result in the studies of geometric phases for neutral particle under the influence of the Lorentz symmetry breaking. By comparing with Ref. [27], we should note that the nature of the geometric phase obtained by Wei *et al* [27] is a consequence of the response to the particle with respect to electric and magnetic field applied, therefore it depends on the polarization which is a characteristic of the particle. However, the phase shift analysed in this work is generated the zero component of the 4-vector background  $b^0$ . This 4-vector generates an anisotropy in the spacetime, thus, if we change the particle, the response of the particle changes, but the geometric phase produced by the 4-vector background remains unchanged. Thereby, if we want to analyse a neutral particle with the settings in the electric and magnetic fields given in [27] plus the Lorentz symmetry violation background in the nonrelativistic limit, we have two independent contributions for the geometric phase. One should expect that the coming phase from the presence of the 4-vector background being weaker than that generated by the polarization of the particle, but there exists this contribution to the geometric phase. For the relativistic Abelian geometric phase (7), we can also expect that this phase is quite small, thus, one should repeat the closed path given in (7) many times in order to obtain a significant contribution to the geometric phase from the 4-vector background. For instance, let us suppose an experimental ability to measure geometrical phases as small as  $10^{-4}\text{rad}$  [64], thus, we can affirm that the theoretical phase induced for a neutral particle can not be larger than this value, that is,  $\nu\lambda B_0 < 10^{-4}\text{rad}$ . In this way, by taking the values of the fields  $|\vec{E}| \approx 10^7 \frac{\text{V}}{\text{m}}$ ,  $B_0 \approx 5\text{T}$ ,  $r_0 = 10^{-5}\text{m}$  (which correspond to usual values of electric fields and radius for 1D mesoscopic rings [65]), and working with the natural units system  $\hbar = c = 1$  (wherein  $1\text{V} = 11.7\text{eV}$ ), we can estimate a upper bound for the constant  $\nu = gb^0$  given by  $\nu < 10^{-12}(\text{eV})^{-3}$ .

At this moment, let us discuss the bound states which arise in the relativistic quantum dynamics of the neutral particle described by the Dirac equation (4) when we restrict the neutral particle to move in a region between two coaxial cylinders with impenetrable walls. Let us take the solution of the Dirac equation (4) in the form:  $\psi = e^{-i\mathcal{E}t} (v \varsigma)^T$ , where  $v$  and  $\varsigma$  are two-spinors. Thus, substituting this solution into (4), we obtain two coupled equations for  $v$  and  $\varsigma$ , where the first coupled equation is

$$\left(\mathcal{E} - m + \frac{\nu\lambda^2}{\rho^2}\right) v = \left[-i\sigma^1 \frac{\partial}{\partial \rho} - \frac{i\sigma^1}{2\rho} - \frac{i\sigma^2}{\rho} \frac{\partial}{\partial \varphi} - i\sigma^3 \frac{\partial}{\partial z} - \frac{\Phi}{2\pi\rho} \sigma^2\right] \varsigma, \quad (8)$$

and the second coupled equation is

$$\left(\mathcal{E} + m - \frac{\nu\lambda^2}{\rho^2}\right)\varsigma = \left[-i\sigma^1\frac{\partial}{\partial\rho} - \frac{i\sigma^1}{2\rho} - \frac{i\sigma^2}{\rho}\frac{\partial}{\partial\varphi} - i\sigma^3\frac{\partial}{\partial z} - \frac{\Phi}{2\pi\rho}\sigma^2\right]v. \quad (9)$$

Thus, by eliminating  $\varsigma$  in the equation (9) and neglecting the terms proportional to  $m^{-2}$  and  $\nu^2$  (we consider  $\nu$  being a small coupling constant), we obtain the following second order differential equation

$$\begin{aligned} \left[\mathcal{E}^2 - m^2 + \frac{2m\nu\lambda^2}{\rho^2}\right]v = & -\frac{\partial^2 v}{\partial\rho^2} - \frac{1}{\rho}\frac{\partial v}{\partial\rho} + \frac{v}{4\rho^2} - \frac{1}{\rho^2}\frac{\partial^2 v}{\partial\varphi^2} - \frac{\partial^2 v}{\partial z^2} + \frac{i\sigma^3}{\rho^2}\frac{\partial v}{\partial\varphi} \\ & + \sigma^3\frac{\Phi}{2\pi\rho^2}v + 2i\frac{\Phi}{2\pi\rho}\frac{\partial v}{\partial\varphi} + \left(\frac{\Phi}{2\pi}\right)^2\frac{v}{\rho^2}. \end{aligned} \quad (10)$$

We can see in Eq. (10) that  $v$  is an eigenfunction of  $\sigma^3$  whose eigenvalues are  $s = \pm 1$ . In that way, we can split  $v$  into  $v = (v_+ \ v_-)^T$  where  $\sigma^3 v_+ = v_+$  and  $\sigma^3 v_- = -v_-$ . Hence, in order to solve the second order differential equation for both components  $v_+$  and  $v_-$ , we write these components in the compact form  $v_s$ , with  $\sigma^3 v_s = s v_s$ . Thus, we can take the solution of the second order differential equation (10) in the form  $v_s = C e^{i(l+\frac{1}{2})\varphi} e^{ikz} R_s(\rho)$ , where  $l$  is a integer number,  $k$  is a constant and  $C$  is a constant. Substituting  $v_s$  into (10), we obtain

$$\left[\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \frac{(\zeta_s^2 - \tau^2)}{\rho^2} + \kappa^2\right]R_s(\rho) = 0, \quad (11)$$

with  $\zeta_s = l + \frac{1}{2}(1-s) - \frac{\Phi}{2\pi}$ ,  $\tau^2 = 2m\nu\lambda^2$  and  $\kappa^2 = \mathcal{E}^2 - m^2 - k^2$ . We have that the equation (11) is the Bessel differential equation and the general solution for (11) is given by  $R_s(\rho) = A_l J_\eta(\kappa\rho) + B_l N_\eta(\kappa\rho)$ , where  $\eta = \sqrt{\gamma_s^2 - \tau^2}$ ,  $J_\eta(\kappa\rho)$  and  $N_\eta(\kappa\rho)$  are the Bessel functions of the first and second kind, respectively. Now, we consider the neutral particle is restricted to move in a region between two coaxial cylindrical surfaces  $\rho = \rho_a$  and  $\rho = \rho_b$ , where  $\rho_b > \rho_a$ . Thus, by considering the boundaries of these regions as impenetrable, we require that the wave function satisfies the boundary conditions:  $R(\rho_a) = R(\rho_b) = 0$ . These conditions provide us the following equation for the energy spectrum of the neutral particle

$$J_\eta(\kappa\rho_a) N_\eta(\kappa\rho_b) - J_\eta(\kappa\rho_b) N_\eta(\kappa\rho_a) = 0. \quad (12)$$

In order to obtain the energy spectrum explicitly, we consider a situation in which  $\kappa\rho_a \gg 1$  and  $\kappa\rho_b \gg 1$ . Then, we apply the well-known asymptotic of the Hankel function, when

$\eta$  is fixed, so

$$J_\eta(\kappa\rho_a) \approx \sqrt{\frac{2}{\pi\kappa\rho_a}} \left[ \cos\left(\kappa\rho_a - \frac{\eta\pi}{2} - \frac{\pi}{4}\right) - \frac{(4\eta^2 - 1)}{8\kappa\rho_a} \sin\left(\kappa\rho_a - \frac{\eta\pi}{2} - \frac{\pi}{4}\right) \right]; \quad (13)$$

$$N_\eta(\kappa\rho_a) \approx \sqrt{\frac{2}{\pi\kappa\rho_a}} \left[ \sin\left(\kappa\rho_a - \frac{\eta\pi}{2} - \frac{\pi}{4}\right) + \frac{(4\eta^2 - 1)}{8\kappa\rho_a} \cos\left(\kappa\rho_a - \frac{\eta\pi}{2} - \frac{\pi}{4}\right) \right],$$

and by interchanging  $\rho_a$  for  $\rho_b$ , we obtain similar expressions for  $J_\eta(\kappa\rho_b)$  and  $N_\eta(\kappa\rho_b)$ . Thus, substituting the expressions (13) and similar expressions for  $J_\eta(\kappa\rho_b)$  and  $N_\eta(\kappa\rho_b)$  into (12), we obtain  $\kappa^2 \approx \left(\frac{n\pi}{\rho_b - \rho_a}\right)^2 + \frac{(4\eta^2 - 1)}{4\rho_a\rho_b}$ , with  $n = 0, 1, 2, 3 \dots$ . Hence, by using the definition of the parameters  $\zeta_s, \kappa, \tau$  given earlier, the energy levels for this system are

$$\mathcal{E}_{n,l}^2 \approx m^2 + k^2 + \frac{(n\pi)^2}{(\rho_b - \rho_a)^2} + \frac{\left[l + \frac{1}{2}(1 - s) - \frac{\Phi}{2\pi}\right]^2}{\rho_a\rho_b} - \frac{2m\nu\lambda^2}{\rho_a\rho_b} - \frac{1}{4\rho_a\rho_b}, \quad (14)$$

The expression (14) corresponds to the relativistic energy levels for bound states when the neutral particle is restrict to move in a confined region between two coaxial cylinders surfaces with impenetrable walls. These bound states depend on the relativistic geometric quantum phase  $\Phi$  given in the expression (7) with periodicity  $\phi_0 = 2\pi$ . Note that the dependence of the energy levels (14) on the relativistic geometric phase  $\Phi$  provide us an analogous effect to the Aharonov-Bohm effect for bound states [5]. To obtain the nonrelativistic expression for the energy levels, we must apply the Taylor expansion in the expression (14), and obtain

$$\mathcal{E}_{n,l} \approx m + \frac{1}{2m} \frac{(n\pi)^2}{(\rho_b - \rho_a)^2} + \frac{1}{2m} \frac{\left[l + \frac{1}{2}(1 - s) - \frac{\Phi}{2\pi}\right]^2}{\rho_a\rho_b} - \frac{\nu\lambda^2}{\rho_a\rho_b} - \frac{1}{8M\rho_a\rho_b} + \frac{k^2}{2m}, \quad (15)$$

where we must remember that  $m$  is the rest mass of neutral particle and the remaining terms of the equation (15) correspond to the nonrelativistic energy levels for bound states. Note that the energy levels for the bound states depend on the quantum flux  $\Phi$  as obtained in [27]. In order to compare with the nonrelativistic energy levels of the bound states given in [27] (where it is considered that the neutral particle moves in a ring of radius  $R$ ), from the expression (14), we can see that when  $\rho_b \rightarrow \rho_a \Rightarrow \mathcal{E} \rightarrow \infty$ . Thus, to obtain the limit where  $\mathcal{E} = \text{const}$  as  $\rho_b \rightarrow \rho_a$ , we must introduce an attractive potential in the region  $\rho_a < \rho < \rho_b$  in order to compensate the energy increase of the radial modes in this limit [66]. In that way, the energy levels (14) becomes

$$\mathcal{E}_l^2 \approx m^2 + k^2 + \frac{\left[l + \frac{1}{2}(1 - s) - \frac{\Phi}{2\pi}\right]^2}{\rho_a^2} - \frac{2m\nu\lambda^2}{\rho_a^2} - \frac{1}{4\rho_a^2}. \quad (16)$$

Taking the nonrelativistic limit of the expression (16) through the Taylor expansion, we obtain

$$\mathcal{E}_l \approx m + \frac{1}{2m} \frac{\left[l + \frac{1}{2}(1-s) - \frac{\Phi}{2\pi}\right]^2}{\rho_a^2} - \frac{\nu\lambda^2}{\rho_a^2} - \frac{1}{8m\rho_a^2}, \quad (17)$$

where  $m$  is the rest mass of the spin-half neutral particle and the remaining terms of the energy levels (17) correspond to the energy levels of an one-dimensional quantum ring with radius  $\rho_a$ . We can also see that we have an analogous expression of that one given in [27], where the energy levels for the bound states depend on the geometric phase obtained by Wei *et al* [27]:  $\phi_{\text{WHW}} = 2\pi\alpha\lambda B_0$ , where  $\alpha$  corresponds to the polarizability. Note that the last term of the expression (17) corresponds to the dynamics of a particle in a two-dimensional surface inside a three-dimensional space as showed in [67].

### III. CONCLUSIONS

In this work, we have proposed a theoretical approach to study a relativistic quantum dynamics of a spin-half neutral particle based on the Lorentz symmetry violation background, where the wave function of the spin-half neutral particle acquires a relativistic Abelian quantum phase analogous to the geometric phase obtained by Wei *et al* [27] for a spinless neutral particle with an induced electric dipole moment in the rest frame of the observers. This approach consists in the introduction of a new term in the Dirac equation in such a way that we can obtain, in the nonrelativistic limit of the Dirac equation, an equation of motion analogous to that one obtained by Wei *et al.* [27] in the study of the geometric phase for a spinless neutral particle with induced electric dipole. We have seen that this approach provides us a relativistic analogue of the Abelian geometric quantum phase for a neutral particle with an induced electric dipole moment, where this relativistic quantum phase is produced by the interaction between a fixed time-like 4-vector background and crossed electric and magnetic fields. Moreover, we have seen that this geometric quantum phase is nondispersive. Furthermore, this approach allows us to obtain the relativistic bound states for a spin-half neutral particle when the neutral particle is confined to moving in a region between two coaxial cylinder shells, showing the dependence of the energy levels on the relativistic geometric phase produced by the Lorentz symmetry violation background. Finally, in the nonrelativistic limit of the energy levels, we have seen that we can obtain a

dependence of the energy levels on the geometric quantum phase analogous to that one given in Ref. [27] for a spinless neutral particle with induced electric dipole. We would like to comment that it is hard to discuss this relativistic approach in the phenomenological context for a spin-half particle as done by Wei *et al.* in [27] for an induced electric dipole since we can expect that the phase shift provided by the Lorentz symmetry violation background is quite small, but we hope that this theoretical approach can provide new discussions about geometric quantum phases for neutral particles, for instance, if it can be extended for discussions about the quantum phase obtained by Spavieri [31–34] for neutral particle with electric dipole moment, and the influence of the Lorentz symmetry breaking on the Landau quantization for neutral particles [41].

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- [68] We have chosen to discuss the case where  $gb^0 = \nu$  because we intend to obtain a nonrelativistic system analogous to that one worked in Ref. [27] from the Lorentz symmetry violation background. In the case  $gb^0 \neq \nu$ , we can see that we can obtain the relativistic Abelian geometric phase, but the nonrelativistic system has two different coupling constants, thus, it differs from that one of Ref. [27]. In this way, it makes sense to take  $\nu = 0$ , and obtain just the geometric phase without discussing the influence of local terms in this quantum dynamics.
- [69] This can be checked through the equation [60]  $\frac{d\hat{\Pi}}{dt} = i \left[ H_D, \hat{\Pi} \right] = \nu \vec{\alpha} \times \vec{B}_{\text{eff}} + \vec{\nabla} (\nu E^2)$ , where  $\frac{\partial \hat{\Pi}}{\partial t} = 0$ ,  $H_D$  is the Dirac Hamiltonian which corresponds to the right-hand-side of the equation (4), and  $\vec{B}_{\text{eff}} = \vec{\nabla} \times (\vec{E} \times \vec{B})$  is the effective magnetic field. With the field configuration considered in Eq. (8), we have that  $\vec{B}_{\text{eff}} = 0$ . Thus, we have that  $\frac{d\hat{\Pi}}{dt} = \vec{\nabla} (\nu E^2)$ .